



An Investigation Into Smarandachely Product Cordial Labeling Of Rooted Product Of Complete Graph $P_n \circ K_m$

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Abstract- This study explores the labeling of the Rooted Product of Complete Graph $P_n \circ K_m$ where $P-\{n\}$ times and $K-\{m\}$ times under the condition of Smarandachely Product Cordial Labeling. In the context of graph labeling, Smarandachely Product Cordial Labeling is defined as a function $f: E(G) \rightarrow \{0, 1\}$ that assigns labels to the edges of graph G with the induced labeling $f(u)f(v)$ on each edge $uv \in E(G)$. The labeling must satisfy the constraints: $|v_f(0) - v_f(1)| \geq 2$ and $|e_f(0) - e_f(1)| \geq 2$, where $v_f(0)$ and $v_f(1)$ represent the number of vertices with labels 0 and 1, respectively, and $e_f(0)$ and $e_f(1)$ denote the number of edges labeled 0 and 1, respectively. The paper aims to provide a detailed examination of this labeling scheme for the specified graph product, assessing its feasibility and implications for graph theory.

Key Words- Cordial labeling, Smarandachely product cordial labeling, Rooted Product of complete Graph.

Introduction- Graph labeling is an increasingly significant area of research with diverse scientific and technological applications. It plays a crucial role in various domains, including radio astronomy, radar and missile guidance system design, X-ray crystallography for material spectral analysis, communication networks, and transportation systems. In the context of graph theory, graph labeling involves assigning integers to the edges, vertices, or both components of a graph, based on specific criteria. This process enables the differentiation of adjacent edges or vertices. The concept of graph labeling was first introduced by Rosa (1967, pp. 349-355).

Smarandachely Product Cordial Labeling, initially introduced by Smarandache (1996, pp.1-2-3), investigates the cordial properties of graphs under product operations. The formal definition of Smarandachely Product Cordial Labeling was later established in 2018 by Patel, Prajapati, and Kansagara (2018, pp. 146-159), specifically focusing on the product cordial labeling of extensions of the Barbell Graph. Their research concentrated on the examination of product cordial labeling for graphs derived from various operations applied to the Barbell Graph.

The present study is now focusing on a distinct class of graph, specifically the Rooted Product of Complete Graphs. Graphs serve as effective models for interconnection networks, where vertices represent processors and edges denote communication links. It introduces a novel interconnection network topology, known as the Rooted Product of Complete Graph, which adheres to the criteria for Smarandachely Product Cordial Labeling. This labeling scheme imposes specific conditions that the graph must satisfy, contributing to its theoretical and practical applications in network design.

Theorem 1: The Rooted Product of complete graph $P_n \circ K_m$ admits Smarandachely product cordial labeling, where $m, n \geq 3$.

Proof:

Case1-When m and n both are even

Steps to construct a graph $G = P_n \circ K_m$:-

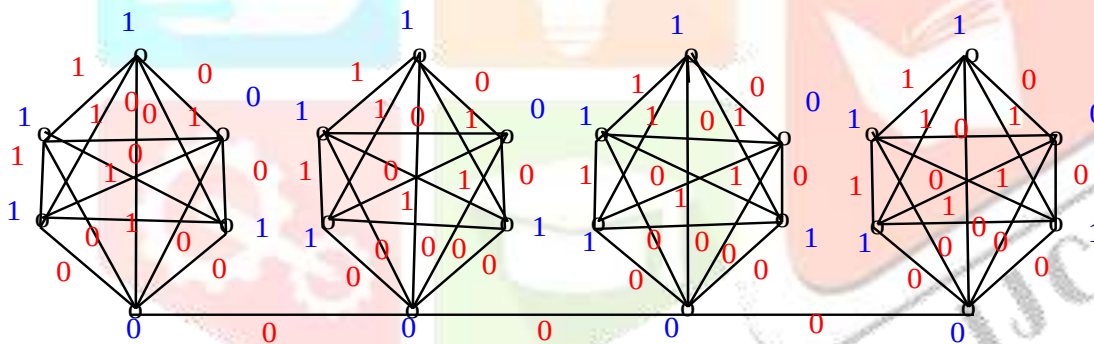
First, we join all the base vertices of P_n and then the rest vertices of K_m join with the base vertices of P_n and with each other and create a cycle and complete graph.

Label the Graph $G = P_n \circ K_m$ in the following steps-

- i. Label all vertices of P_n with (0) respectively.
- ii. Label all vertices of K_m clock wise with (1,1,1,0,1,1,0.....)respectively.

In graph no. 1.1 Graph $P_4 \circ K_6$ is labeled as follows: -

- i. Label all vertices of P_4 with (0) respectively.
- ii. Label all vertices of K_6 clock wise with (1, 1, 1, 0, 1, 1) respectively.



Graph No. 1.1: Smaradachely product Cordial Labeling of $P_4 \circ K_6$ Graph

$$|\sum v_f(0) - \sum v_f(1)| \geq 2$$

$$|8 - 16| \geq 2$$

$$8 \geq 2$$

$$|\sum e_f(0) - \sum e_f(1)| \geq 2$$

$$|39 - 24| \geq 2$$

$$15 \geq 2$$

Sub case I- When $m, n=4, 6, 8, 10, \dots$

We apply the condition of Smarandachely product Cordial labeling –

$$|v_f(0) - v_f(1)| \geq 2 \text{ and } |e_f(0) - e_f(1)| \geq 2$$

1.1 When $m=4$, $n=4, 6, 8, 10, \dots$

$$\{v(1)=3n+6\}$$

$$\{v(0)=n+2\}$$

$$\{e(1)=3n+6\}$$

$$\{e(0)=4n+7\}$$

$$|v_f(0)-v_f(1)| \geq 2$$

$$|(n+2)-(3n+6)| \geq 2$$

$$|n-3n+2-6| \geq 2$$

$$|-2n-4| \geq 2$$

$$2n+4 \geq 2$$

Where $n=2, 4, 6, 8, \dots$

$$|e_f(0) - e_f(1)| \geq 2$$

$$|(4n+7)-(3n+6)| \geq 2$$

$$|4n-3n+7-6| \geq 2$$

$$|n+1| \geq 2$$

$$n+1 \geq 2$$

Where $n=2, 4, 6, 8, \dots$

1.2 When $m=6$, $n=4, 6, 8, 10, \dots$

$$\{v(1)=8n+8\}$$

$$\{v(0)=4n+4\}$$

$$\{e(1)=12n+12\}$$

$$\{e(0)=20n+19\}$$

$$|v_f(0)-v_f(1)| \geq 2$$

$$|(4n+4)-(8n+8)| \geq 2$$

$$|4n-8n+4-8| \geq 2$$

$$|-4n-4| \geq 2$$

$$4n+4 \geq 2$$

Where $n=1, 2, 3, 4, 5, 6, \dots$

$$|e_f(0) - e_f(1)| \geq 2$$

$$|(20n+19)-(12n+12)| \geq 2$$

$$|20n-12n+19-12| \geq 2$$

$$|8n+7| \geq 2$$

$$8n+7 \geq 2$$

Where $n=1, 2, 3, 4, 5, 6, \dots$

1.3 When $m=8$, $n=4, 6, 8, 10, \dots$

$$\{v(1)=12n+12\}$$

$$\{v(0)=4n+4\}$$

$$\{e(1)=30n\}$$

$$\{e(0)=28n-1\}$$

$$|v_f(0)-v_f(1)| \geq 2$$

$$|(4n+4)-(12n+12)| \geq 2$$

$$|4n-12n+4-12|\geq 2$$

$$|-8n-8|\geq 2$$

$$8n+8\geq 2$$

Where $n=1, 2, 3, 4, 5, 6, \dots$

$$|e_f(0) - e_f(1)|\geq 2$$

$$|(28n-1)-(30n)|\geq 2$$

$$|28n-30n-1|\geq 2$$

$$|-2n-1|\geq 2$$

$$2n+1\geq 2$$

Where $n=2, 3, 4, 5, 6, \dots$

1.4 When $m=10, n=4, 6, 8, 10, \dots$

$$\{v(1)=14n\}$$

$$\{v(0)=6n\}$$

$$\{e(1)=42n\}$$

$$\{e(0)=50n-1\}$$

$$|v_f(0)-v_f(1)|\geq 2$$

$$|6n-14n|\geq 2$$

$$|-8n|\geq 2$$

$$8n\geq 2$$

Where $n=2, 3, 4, 5, 6, \dots$

$$|e_f(0) - e_f(1)|\geq 2$$

$$|(50n-1)-(42n)|\geq 2$$

$$|50n-42n-1|\geq 2$$

$$|8n-1|\geq 2$$

$$8n-1\geq 2$$

Where $n=2, 3, 4, 5, 6, \dots$

1.5 When $m=12, n=4, 6, 8, 10, \dots$

$$\{v(1)=18n\}$$

$$\{v(0)=6n\}$$

$$\{e(1)=72n\}$$

$$\{e(0)=62n-1\}$$

$$|v_f(0)-v_f(1)|\geq 2$$

$$|6n-18n|\geq 2$$

$$|-12n|\geq 2$$

$$12n\geq 2$$

Where $n=2, 3, 4, 5, 6, \dots$

$$|e_f(0) - e_f(1)|\geq 2$$

$$|(62n-1)-(72n)|\geq 2$$

$$|62n-72n-1|\geq 2$$

$$|-10n-1|\geq 2$$

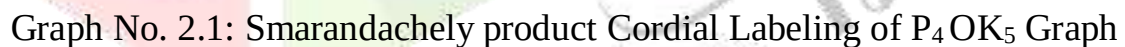
$$10n+1\geq 2$$

Case2-When n is even and m is odd-

First, we join all the base vertices of P_n and then the rest vertices of K_m join with the base vertices of P_n and with each other and create a cycle and complete graph.

- i. Label all vertices of P_n with (0) respectively.
- ii. Label all vertices of K_m clock wise with (1,1,1,0,1,1,0.....) respectively.

- Label all vertices of P_4 with (0) respectively.
- Label all vertices of K_5 clock wise with (1, 1, 1, 0) respectively.



2.1 When $m=5$, $n=4, 6, 8, 10, \dots$

$$\begin{aligned}\{v(1) &= 6n\} \\ \{v(0) &= 4n\} \\ \{e(1) &= 6n\} \\ \{e(0) &= 16n-1\}\end{aligned}$$

$$\begin{aligned}|v_f(0) - v_f(1)| &\geq 2 \\ |4n - 6n| &\geq 2 \\ |-2n| &\geq 2 \\ 2n &\geq 2\end{aligned}$$

Where $n=2, 3, 4, 5, 6, \dots$

$$\begin{aligned}|e_f(0) - e_f(1)| &\geq 2 \\ |(16n-1) - (6n)| &\geq 2 \\ |16n - 6n - 1| &\geq 2 \\ |10n - 1| &\geq 2 \\ 10n - 1 &\geq 2\end{aligned}$$

Where $n=2, 3, 4, 5, 6, \dots$

2.2 When $m=7, n=4, 6, 8, 10, \dots$

$$\begin{aligned}\{v(1) &= 10n\} \\ \{v(0) &= 4n\} \\ \{e(1) &= 20n\} \\ \{e(0) &= 24n-1\}\end{aligned}$$

$$\begin{aligned}|v_f(0) - v_f(1)| &\geq 2 \\ |4n - 10n| &\geq 2 \\ |-6n| &\geq 2 \\ 6n &\geq 2\end{aligned}$$

Where $n=2, 3, 4, 5, 6, \dots$

$$\begin{aligned}|e_f(0) - e_f(1)| &\geq 2 \\ |(24n-1) - (20n)| &\geq 2 \\ |24n - 20n - 1| &\geq 2 \\ |4n - 1| &\geq 2 \\ 4n - 1 &\geq 2\end{aligned}$$

Where $n=2, 3, 4, 5, 6, \dots$

2.3 When $m=9, n=4, 6, 8, 10, \dots$

$$\begin{aligned}\{v(1) &= 12n\} \\ \{v(0) &= 6n\} \\ \{e(1) &= 30n\} \\ \{e(0) &= 44n-1\}\end{aligned}$$

$$\begin{aligned}|v_f(0) - v_f(1)| &\geq 2 \\ |6n - 12n| &\geq 2 \\ |-6n| &\geq 2 \\ 6n &\geq 2\end{aligned}$$

Where $n=2, 3, 4, 5, 6, \dots$

$$\begin{aligned}
 |e_f(0) - e_f(1)| &\geq 2 \\
 |(44n-1)-(30n)| &\geq 2 \\
 |44n-30n-1| &\geq 2 \\
 |14n-1| &\geq 2 \\
 14n-1 &\geq 2
 \end{aligned}$$

Where $n=2, 3, 4, 5, 6, \dots$

Case3-When n is odd and m is odd-

Steps to construct a graph $G = P_n \circ K_m$:-

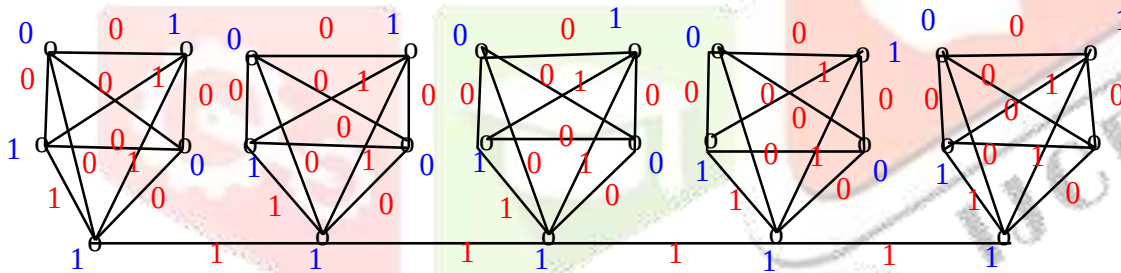
First, we join all the base vertices of P_n and then the rest vertices of K_m join with the base vertices of P_n and with each other and create a cycle and complete graph.

Label the Graph $G = P_n \circ K_m$ in the following steps-

- Label all vertices of P_n with (1) respectively.
- Label all vertices of K_m anti clock wise with (1, 0, 1, 0, 1, 0.....) respectively.

In graph no. 3.1 Graph $P_5 \circ K_5$ is labeled as follows: -

- Label all vertices of P_4 with (1) respectively.
- Label all vertices of K_5 anti clock wise with (1, 0, 1, 0) respectively.



Graph No. 3.1: Smarandachely product Cordial Labeling of $P_5 \circ K_5$ Graph

$$\begin{aligned}
 |\sum v_f(0) - \sum v_f(1)| &\geq 2 \\
 |10-15| &\geq 2 \\
 5 &\geq 2
 \end{aligned}$$

$$\begin{aligned}
 |\sum e_f(0) - \sum e_f(1)| &\geq 2 \\
 |21-11| &\geq 2 \\
 10 &\geq 2
 \end{aligned}$$

Sub case III- When $m=5,7,9,11, \dots$, $n=3,5,7,9,11, \dots$

We apply the condition of Smarandachely product Cordial labeling –

$$|v_f(0)-v_f(1)| \geq 2 \text{ and } |e_f(0) - e_f(1)| \geq 2$$

3.1 When $m=5$ and $n=3, 5, 7, 9, \dots$

$$\{v(1)=3n\}$$

$$\{v(0)=2n\}$$

$$\{e(1)=8n+3\}$$

$$\{e(0)=14n+7\}$$

$$|v_f(0)-v_f(1)| \geq 2$$

$$|2n-3n| \geq 2$$

$$|-n| \geq 2$$

$$n \geq 2$$

Where $n=3, 5, 7, 9, \dots$

$$|e_f(0) - e_f(1)| \geq 2$$

$$|(14n+7)-(8n+3)| \geq 2$$

$$|14n-8n+7-3| \geq 2$$

$$|6n+4| \geq 2$$

$$6n+4 \geq 2$$

Where $n=1, 2, 3, 4, 5, 6, \dots$

3.2 When $m=7$ and $n=3, 5, 7, 9, \dots$

$$\{v(1)=8n+4\}$$

$$\{v(0)=6n+3\}$$

$$\{e(1)=12n+5\}$$

$$\{e(0)=32n+16\}$$

$$|v_f(0)-v_f(1)| \geq 2$$

$$|(6n+3)-(8n+4)| \geq 2$$

$$|6n-8n+3-4| \geq 2$$

$$|-2n-1| \geq 2$$

$$2n+1 \geq 2$$

Where $n=1, 2, 3, 4, \dots$

$$|e_f(0) - e_f(1)| \geq 2$$

$$|(32n+16)-(12n+5)| \geq 2$$

$$|32n-12n+16-5| \geq 2$$

$$|20n+11| \geq 2$$

$$20n+11 \geq 2$$

Where $n=1, 2, 3, 4, 5, \dots$

3.3 When $m=9$ and $n=3, 5, 7, 9, \dots$

$$\{v(1)=10n+5\}$$

$$\{v(0)=8n+4\}$$

$$\{e(1)=22n+10\}$$

$$\{e(0)=52n+26\}$$

$$|v_f(0)-v_f(1)| \geq 2$$

$$|(8n+4)-(10n+5)| \geq 2$$

$$|8n-10n+4-5| \geq 2$$

$$|-2n-1| \geq 2$$

$$2n+1 \geq 2$$

Where $n=1, 2, 3, 4, \dots$

$$|e_f(0) - e_f(1)| \geq 2$$

$$|(52n+26)-(22n+10)| \geq 2$$

$$|52n-22n+26-10| \geq 2$$

$$|30n+16| \geq 2$$

$$30n+16 \geq 2$$

Where $n=1, 2, 3, 4, 5, \dots$

3.4 When $m=11$ and $n=3, 5, 7, 9, \dots$

$$\{v(1)=12n+6\}$$

$$\{v(0)=10n+5\}$$

$$\{e(1)=32n+15\}$$

$$\{e(0)=80n+40\}$$

$$|v_f(0)-v_f(1)| \geq 2$$

$$|(10n+5)-(12n+6)| \geq 2$$

$$|10n-12n+5-6| \geq 2$$

$$|-2n-1| \geq 2$$

$$2n+1 \geq 2$$

Where $n=1, 2, 3, 4, \dots$

$$|e_f(0) - e_f(1)| \geq 2$$

$$|(80n+40)-(32n+15)| \geq 2$$

$$|80n-32n+40-15| \geq 2$$

$$|48n+25| \geq 2$$

$$48n+25 \geq 2$$

Where $n=1, 2, 3, 4, 5, \dots$

3.5 When $m=13$ and $n=3, 5, 7, 9, \dots$

$$\{v(1)=14n+7\}$$

$$\{v(0)=12n+6\}$$

$$\{e(1)=44n+21\}$$

$$\{e(0)=114n+57\}$$

$$|v_f(0)-v_f(1)| \geq 2$$

$$|(12n+6)-(14n+7)| \geq 2$$

$$|12n-14n+6-7| \geq 2$$

$$|-2n-1| \geq 2$$

$$2n+1 \geq 2$$

Where $n=1, 2, 3, 4, \dots$

$$|e_f(0) - e_f(1)| \geq 2$$

$$|(114n+57)-(44n+21)| \geq 2$$

$$|114n-44n+57-21| \geq 2$$

$$|70n+36| \geq 2$$

$$70n+36 \geq 2$$

Where $n=1, 2, 3, 4, 5, \dots$

Case4-When n is odd and m is even-

Steps to construct a graph $G = P_n \circ K_m$:-

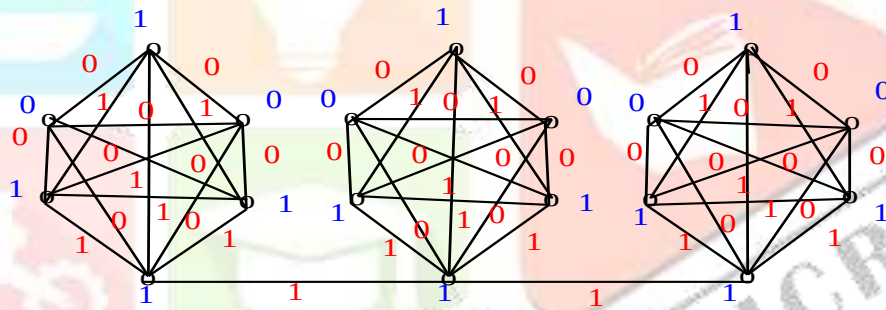
First, we join all the base vertices of P_n and then the rest vertices of K_m join with the base vertices of P_n and with each other and create a cycle and complete graph.

Label the Graph $G = P_n \circ K_m$ in the following steps-

- Label all vertices of P_n with (1) respectively.
- Label all vertices of K_m clock wise with (1, 0, 1, 0, 1, 0.....) respectively.

In graph no. 3.1 Graph $P_3 \circ K_6$ is labeled as follows: -

- Label all vertices of P_4 with (1) respectively.
- Label all vertices of K_5 clock wise with (1, 0, 1, 0, 1, 0) respectively.



Graph No. 4.1: Smarandachely product Cordial Labeling of $P_3 \circ K_6$ Graph

$$|\sum v_f(0) - \sum v_f(1)| \geq 2$$

$$|6-12| \geq 2$$

$$6 \geq 2$$

$$|\sum e_f(0) - \sum e_f(1)| \geq 2$$

$$|27-20| \geq 2$$

$$7 \geq 2$$

Sub case IV- When $m = 4, 6, 8, 10, \dots$ and $n = 3, 5, 7, 9, \dots$

We apply the condition of Smarandachely product Cordial labeling –

$$|v_f(0) - v_f(1)| \geq 2 \text{ and } |e_f(0) - e_f(1)| \geq 2$$

4.1 When $m=4$ and $n=3, 5, 7, 9, \dots$

$$\{v(1) = 6n+3\}$$

$$\{v(0) = 2n+1\}$$

$$\{e(1) = 8n+3\}$$

$$\{e(0)=6n+3\}$$

$$|v_f(0)-v_f(1)| \geq 2$$

$$|(2n+1)-(6n+3)| \geq 2$$

$$|2n-6n+1-3| \geq 2$$

$$|-4n-2| \geq 2$$

$$4n+2 \geq 2$$

Where $n=1, 2, 3, 4, 5, \dots$

$$|e_f(0) - e_f(1)| \geq 2$$

$$|(6n+3)-(8n+3)| \geq 2$$

$$|6n-8n+3-3| \geq 2$$

$$|-2n| \geq 2$$

$$2n \geq 2$$

Where $n=1, 2, 3, 4, 5, 6, \dots$

4.2 When $m=6$ and $n=3, 5, 7, 9, \dots$

$$\{v(1)=8n+4\}$$

$$\{v(0)=4n+2\}$$

$$\{e(1)=14n+6\}$$

$$\{e(0)=18n+9\}$$

$$|v_f(0)-v_f(1)| \geq 2$$

$$|(4n+2)-(8n+4)| \geq 2$$

$$|4n-8n+2-4| \geq 2$$

$$|-4n-2| \geq 2$$

$$4n+2 \geq 2$$

Where $n=1, 2, 3, 4, 5, \dots$

$$|e_f(0) - e_f(1)| \geq 2$$

$$|(18n+9)-(14n+6)| \geq 2$$

$$|18n-14n+9-6| \geq 2$$

$$|4n+3| \geq 2$$

$$4n+3 \geq 2$$

Where $n=1, 2, 3, 4, 5, 6, \dots$

4.3 When $m=8$ and $n=3, 5, 7, 9, \dots$

$$\{v(1)=10n+5\}$$

$$\{v(0)=6n+3\}$$

$$\{e(1)=22n+10\}$$

$$\{e(0)=36n+18\}$$

$$|v_f(0)-v_f(1)| \geq 2$$

$$|(6n+3)-(10n+5)| \geq 2$$

$$|6n-10n+3-5| \geq 2$$

$$|-4n-2| \geq 2$$

$$4n+2 \geq 2$$

Where $n=1, 2, 3, 4, 5, \dots$

$$|e_f(0) - e_f(1)| \geq 2$$

$$|(36n+18)-(22n+10)| \geq 2$$

$$|36n-22n+18-10| \geq 2$$

$$|14n+8| \geq 2$$

$$14n+8 \geq 2$$

Where $n=1, 2, 3, 4, 5, 6, \dots$

4.4 When $m=10$ and $n=3, 5, 7, 9, \dots$

$$\{v(1)=12n+6\}$$

$$\{v(0)=8n+4\}$$

$$\{e(1)=32n+15\}$$

$$\{e(0)=60n+30\}$$

$$|v_f(0)-v_f(1)| \geq 2$$

$$|(8n+4)-(12n+6)| \geq 2$$

$$|8n-12n+4-6| \geq 2$$

$$|-4n-2| \geq 2$$

$$4n+2 \geq 2$$

Where $n=1, 2, 3, 4, 5, \dots$

$$|e_f(0) - e_f(1)| \geq 2$$

$$|(60n+30)-(32n+15)| \geq 2$$

$$|60n-32n+30-15| \geq 2$$

$$|28n+15| \geq 2$$

$$28n+15 \geq 2$$

Where $n=1, 2, 3, 4, 5, 6, \dots$

4.5 When $m=12$ and $n=3, 5, 7, 9, \dots$

$$\{v(1)=14n+7\}$$

$$\{v(0)=10n+5\}$$

$$\{e(1)=44n+21\}$$

$$\{e(0)=90n+45\}$$

$$|v_f(0)-v_f(1)| \geq 2$$

$$|(10n+5)-(14n+7)| \geq 2$$

$$|10n-14n+5-7| \geq 2$$

$$|-4n-2| \geq 2$$

$$4n+2 \geq 2$$

Where $n=1, 2, 3, 4, 5, \dots$

$$|e_f(0) - e_f(1)| \geq 2$$

$$|(90n+45)-(44n+21)| \geq 2$$

$$|90n-44n+45-21| \geq 2$$

$$|46n+24| \geq 2$$

$$46n+24 \geq 2$$

Where $n=1, 2, 3, 4, 5, 6, \dots$

Hence the Rooted product of Complete Graph P_n o K_m admits Smarandachely product cordial labeling, where $m, n \geq 3$.

Hence proved.

Conclusion- Utilizing Smarandachely Product Cordial Labeling, this study examines the structural properties of graphs, revealing new correlations and patterns within the Rooted Product of Complete Graphs. This investigation deepens the understanding of the complex interactions in graph structures and contributes to the broader field of graph theory. It also lays a foundation for further research and encourages scholars to build upon these findings. Specifically, the research applies Smarandachely Product Cordial Labeling to the Rooted Product of Complete Graphs, where the labeling function $f: E(G) \rightarrow \{0,1\}$ is used for the edges of G , with the induced labeling $f(u)f(v)$ on each edge $uv \in E(G)$ satisfying the constraints $|v_f(0) - v_f(1)| \geq 2$ and $|e_f(0) - e_f(1)| \geq 2$.

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