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How Fundamental Theorem Of Arithmetic Was Saved By The Transition From Element-Based Arithmetic To Ideal Based Arithmetic.

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Abstract: Fundamental theorem of arithmetic is a well-known theorem in mathematics especially in number theory which describes that "every integer greater than one is either a prime number or can be represented as a unique product of prime numbers, ignoring the order in which those factors are written". since the fundamental theorem of algebra failed in rings like $\mathbb{Z}[\sqrt{-5}]$ but it was not an end. later, researchers and mathematicians started trying to prove that elements were not the building blocks but ideals were.

Index Terms - Fundamental theorem of arithmetic, elements, ideal numbers.

I. INTRODUCTION

Elementary Number Theory is a branch of mathematics focused on the properties and relationships of integers, primarily studied using arithmetic and basic algebra without relying on advanced calculus of complex analysis. It is widely considered one of the most elegant and accessible areas of mathematics, famous for its tangible problems and deep, sometimes still unsolved mysteries.

The Elementary Number Theory may be said to have begun in ancient Greece, especially around the 6th century BC. Elementary Number Theory can be classified into various branches such as Arithmetic Functions, Diophantine Equations, Combinatorial and Computational Number Theory. Arithmetic Number Theory studies the properties and relationships of Whole Numbers, specifically Integers and Natural Numbers. Combinatorial and Computational Number Theory investigates properties of integers and discrete structures using, respectively, combinatorial arrangements and algorithmic processes. Recent changes in Elementary Number Theory revolve around computational formalization rather than shifts in core theorems. In this paper, we mainly focus on Fundamental Theorem of Arithmetic.

II. STATEMENT OF FUNDAMENTAL THEOREM OF ARITHMETIC

The Fundamental Theorem of Arithmetic makes two very important promises; one is Existence and second is Uniqueness. Any whole number larger than 1 can be broken down into prime factors. A prime number is greater than 1 that can only be divided evenly by 1 and itself, for example: 2, 3, 5, 7, 11, etc.

If a number isn't prime (which makes it a composite number), you can write it as primes multiplied together. Take it for example 12, it's not a prime but we can write it as $2 \times 2 \times 3$. Again take 45: we can write it as $3 \times 3 \times 5$.

Now, uniqueness, which is the cooler part of the theorem, it says that for any given number, there is only one "recipe" of prime numbers that multiplies to it. You can change the order you write them in, but the actual ingredients never change. For example, 12 can be written as $2 \times 2 \times 3$ or $2 \times 3 \times 2$ or $3 \times 2 \times 2$. But no matter what you must use exactly two 2's and one 3's. you can never make 12 using any other combination of primes (like using 5 or 7).

In mathematical terms, we often write this unique prime factorization using exponents, listing the primes in order from smallest to largest:

$$n = p_1^{a_1} p_2^{a_2} p_3^{a_3} \dots p_k^{a_k} \quad (1)$$

where, $p_1 < p_2 < p_3 \dots < p_k$ are distinct prime numbers and the exponents 'ai' are positive integers. For example: $12 = 2^2 \times 3^1$; $360 = 2^3 \times 3^2 \times 5^1$.

II. (1) Failure of Fundamental Theorem of Arithmetic

The failure of Fundamental Theorem of Arithmetic in rings like $\mathbb{Z}[\sqrt{-5}]$ was not a dead end for mathematics; rather, it was the catalyst for the development of Algebraic Number Theory. When Ernst Kummer and later Richard Dedekind realized that elements in these rings refused to factor uniquely, they didn't conclude that the math was broken. Instead, they concluded that the elements were not the fundamental building blocks - the Ideals were.

Consider the ring of integers $\mathbb{Z}[\sqrt{-5}]$. In this domain, the number 6 can be factored in two distinct ways:

$$6 = 2 \times 3$$

$$6 = (1 + \sqrt{-5})(1 - \sqrt{-5})$$

The elements 2, 3, $1 + \sqrt{-5}$ and $1 - \sqrt{-5}$ are all irreducible in $\mathbb{Z}[\sqrt{-5}]$, and none are associates of one another (they do not differ merely by a unit factor). Because, $2 \neq \pm(1 + \sqrt{-5})$ and $2 \neq \pm(1 - \sqrt{-5})$, the ring $\mathbb{Z}[\sqrt{-5}]$ lacks the Unique Factorization Property. This behavior meant that the notion of a "Prime Element" (an element p , where $p \mid ab \Rightarrow p \mid a$ or $p \mid b$) failed, rendering the traditional Fundamental Theorem of Arithmetic useless in these generalized contexts.

II. (2) Impact of failure of Fundamental Theorem of Arithmetic

The failure of Fundamental Theorem of Arithmetic introduces significant complications but also drives deeper mathematical developments:

Loss of Predictability in Diophantine Equations:

In Number Theory, Unique Factorization is routinely used to solve Polynomial Equations over integers. When Factorization fails, standard techniques- such as equating greatest common divisor or matching prime powers- collapse, making it notoriously difficult to analyze Ideals and solutions to Diophantine Equations.

The shift to Ideal Theory (Dedekind Domains):

Because elements fail to factor uniquely nineteenth century Mathematicians (such as Richard Dedekind) resolved the crisis by shifting focus from elements to ideals. Although element factorization fails in rings like $\mathbb{Z}[\sqrt{-5}]$, ideal factorization into prime ideals remains unique if the ring is a Dedekind Domain. For Example, while the Principal Ideal $\langle 6 \rangle$ does not factor uniquely into Principal Prime Ideals, it factors uniquely as a product of Non-Principle Prime Ideals:

$$\langle 6 \rangle = \langle 2, 1 + \sqrt{-5} \rangle^2 \langle 3, 1 + \sqrt{-5} \rangle \langle 3, 1 - \sqrt{-5} \rangle$$

III. Transition from Elements to Ideals

The transition of Fundamental Theorem of Arithmetic from Elements to Ideals was done due to the hard work by Ernst Kummer, who invented "Ideal Numbers" to recover Unique Factorization in cyclotomic fields. After this, Richard Dedekind later formalized these into the modern notion of an ideal.

The paradigm shift was radical: instead of focusing on the elements of the ring, Mathematicians shift their focus to the ideals generated by those elements.

3.1 Redefining Divisibility:

In the ring $\mathcal{O}_K$, a Principle Ideal (a) is the set of all multiples of a . Divisibility is redefined as containment:

$$a \mid b \iff (b) \subseteq (a)$$

While the elements $6, 2, 3, (1 + \sqrt{-5})$ and $(1 - \sqrt{-5})$ fail to factor uniquely, the ideals they generate factor uniquely into Prime Ideals.

3.2 Resolving the counter example:

In $\mathbb{Z}[\sqrt{-5}]$, let us define the following ideals:

$$p = (2, 1 + \sqrt{-5})$$

$$q = (3, 1 + \sqrt{-5})$$

$$q' = (3, 1 - \sqrt{-5})$$

These are Prime Ideals. When we perform the factorization of the ideal generated by 6, we get:

$$(6) = p^2 q q'$$

Now, consider the factorizations from above:

$$(2) = p^2$$

$$(3) = q q'$$

$$(1 + \sqrt{-5}) = p q$$

$$(1 - \sqrt{-5}) = p q'$$

The product of the ideals $(2)(3)$ is $p^2 q q'$. The product of the ideals $(1 + \sqrt{-5})(1 - \sqrt{-5})$ is also $(p q)(p q') = p^2 q q'$. The uniqueness is restored at the level of Prime Ideals.

IV. Mathematical Formalization: The Dedekind Domain

From element-centered arithmetic to ideal-centered arithmetic shifted how number theory unfolded. A Dedekind domain can be described as an integral domain, where every non-zero proper ideal decomposes into a product of prime ideals in a unique way.

Through demonstrating the ring of integers $\mathcal{O}_K$ of every number field is a Dedekind domain, researchers showed the Fundamental Theorem of Arithmetic had not vanished, only broadened. Numbers ceased serving as building blocks; distinct subsets within the ring took their place as irreducible elements.

V. Conclusion

The transition from element based to ideal based arithmetic was one of the most consequential shifts in 19th century Mathematics. By moving from the individual element to the ideal, Mathematicians preserved the core utility of the Fundamental Theorem of Arithmetic. This shift did not just patch a hole in the theory; it provided the structural framework for Class Field Theory and Modern Algebraic Geometry. The rescue of the Fundamental Theorem of Arithmetic proved that while unique factorization might fail for the buildings blocks, we can see (elements), it remains an immutable truth for the structures they create (ideals).

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